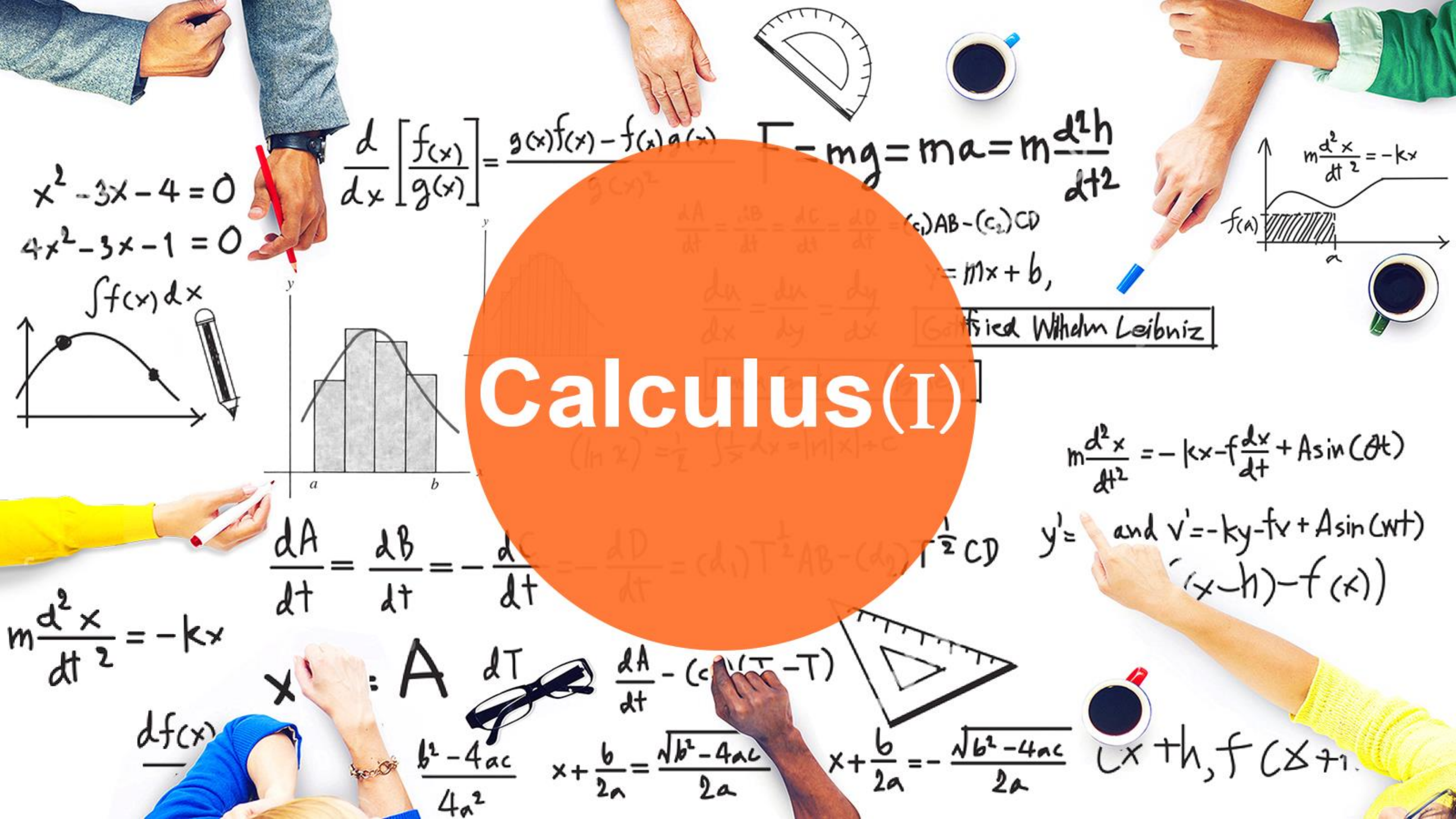
Calculus(Ⅰ)



x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
m d²x/dt² = -kx
y = mx + b,
m d²x/dt² = -kx - f dx/dt + Asin(θt)
and v' = -ky - fv + Asin(wt)

3.6 The Mean Value Theorem for Derivatives

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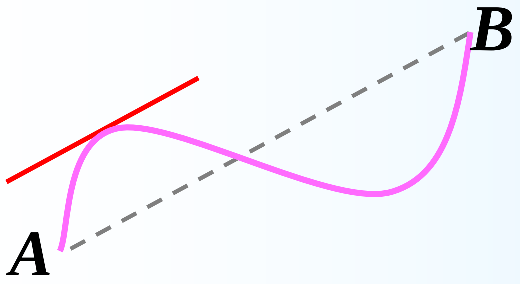
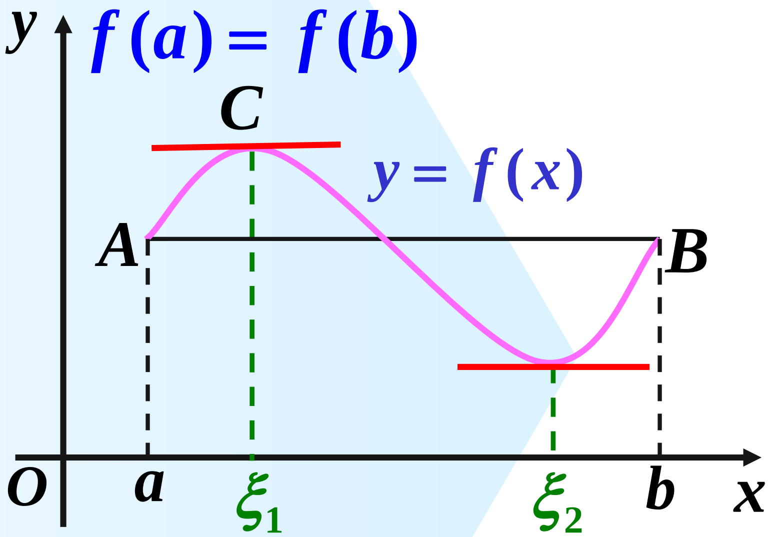
Problem Introduction

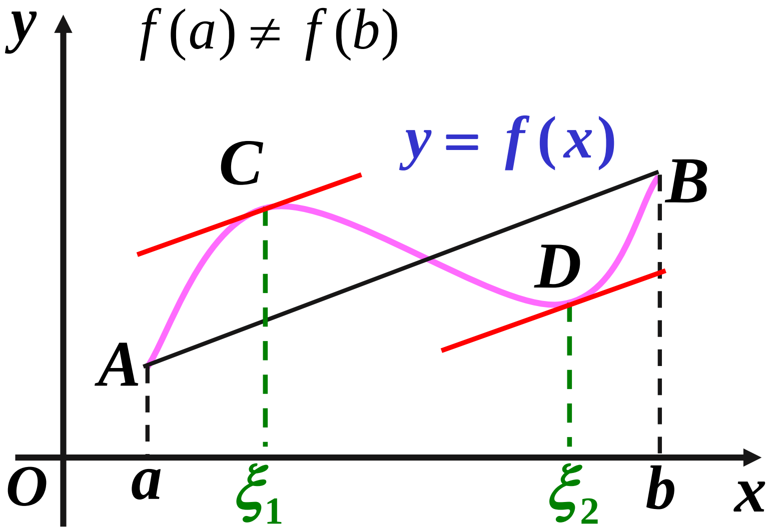
Geometrical fact:





Horizontal Tangent line

$f'(\xi) = 0$

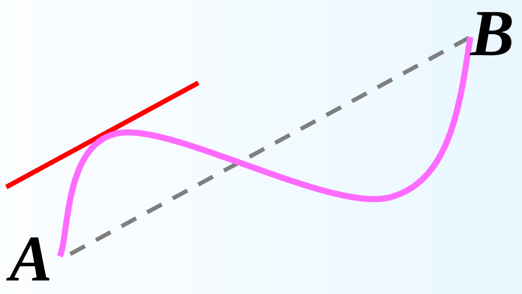


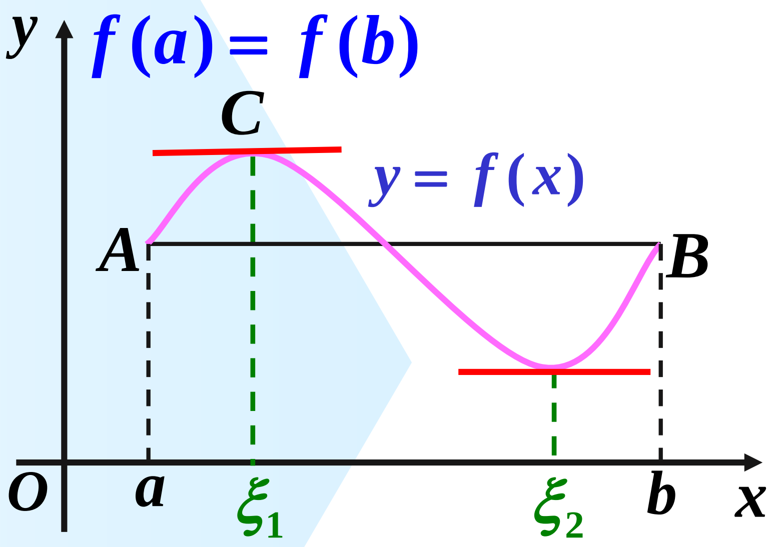
$f'(\xi) = 0$ ✗

$f'(\xi) = ?$

Problem Introduction (L Theorem)

Geometrical fact:

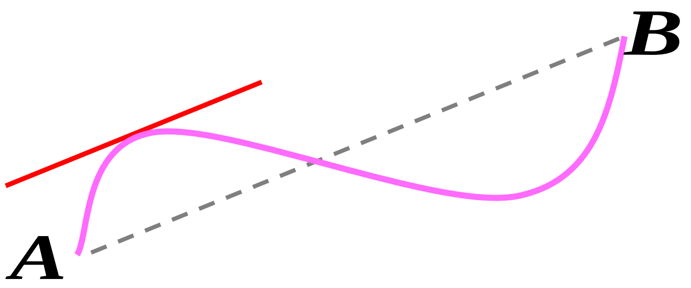


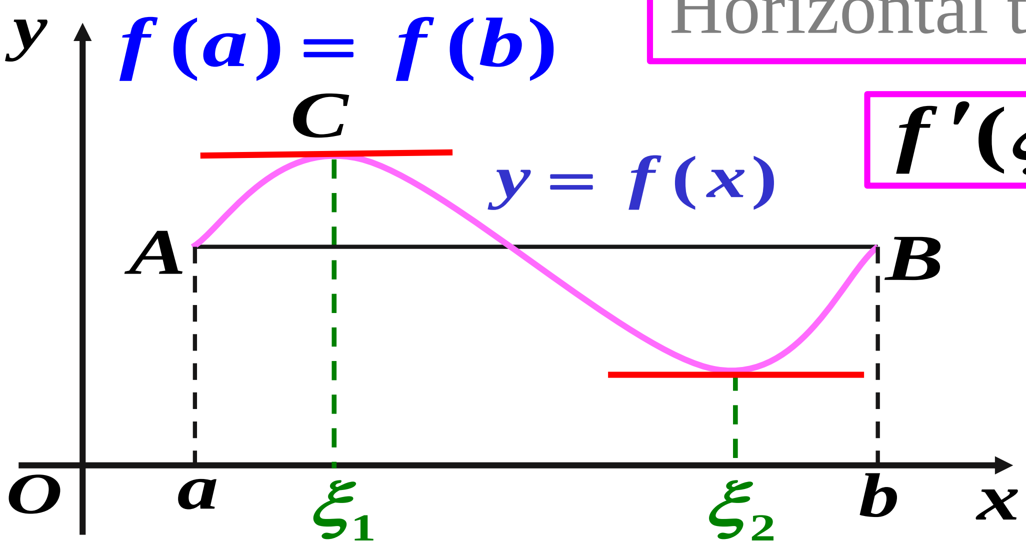


Horizontal Tangent line

$$f'(\xi) = 0$$

Fact: Rolle





Horizontal tangent line

$$f'(\xi) = 0$$

Rolle,(French)1652-1719

Lemma: Rolle Theorem

If $f(x)$: (1) $f(x) \in C[a,b]$;

(2) $f(x)$ is derivative in (a,b);

(3) $f(a) = f(b)$,

$\exists \xi \in (a,b)$ such that $f'(\xi) = 0.$

$f(x) = x^2 - 2x - 3 = (x-3)(x+1).$

$f(x) \in C[-1,3]$, derivative in $(-1,3)$, and $f(-1) = f(3) = 0$,

$f'(x) = 2(x-1)$, take $\xi = 1, (1 \in (-1,3))$ $f'(\xi) = 0.$

Rolle

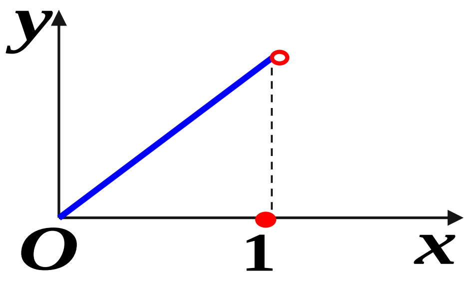
If $f(x)$:

(1) $f(x) \in C[a,b]$;

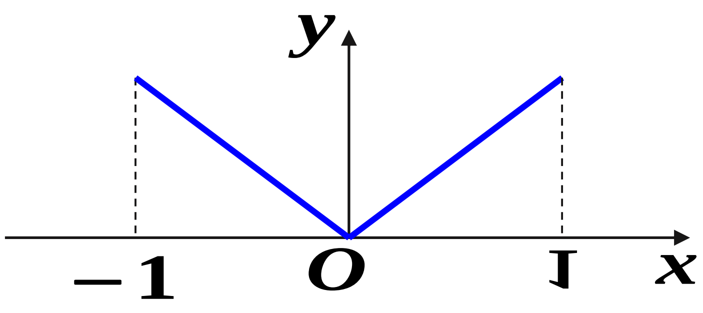
(2) $f(x)$ is derivative in (a,b);

(3) $f(a) = f(b)$,

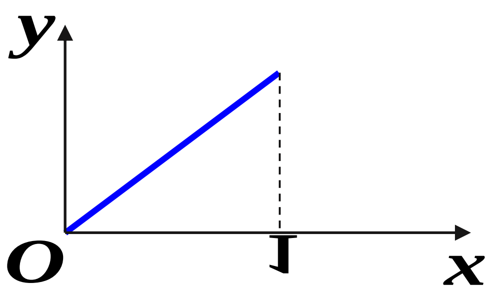
(1) If condition is not enough , then the result may be wrong.



$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$



$$f(x) = |x|,\ x \in [-1,1]$$



$$f(x) = x,\ x \in [0,1]$$

Examples

As for $f(x)=x^3+4x^2-7x-10$, on the $[-1,2]$

Check the rightness of the Rolle Theorem.

(1) Conditions: $f(x)$ in continuous on $[-1,2]$, derivartive in $(-1,2)$,

$$f(-1)=0 = f(2)$$

(2) Result is right. Equation $f'(x)=0$,

Namely, $3x^2+8x-7=0$ has real root.

$$x_1 = \frac{1}{3}(-4-\sqrt{37}), \quad x_2 = \frac{1}{3}(-4+\sqrt{37}) \quad \text{and} \quad x_2 \in (-1,2).$$

Rolle TH: The existence of ξ, It is not necessary to know the value of ξ .

Examples

Prove $x^5 - 5x + 1 = 0$ has one and only one less than 1 and positive root.

(1) Existence

Let $f(x) = x^5 - 5x + 1$, then $f(x)$ is continuous on $[0,1]$,

and $f(0) = 1$, $f(1) = -3$.

By Zero Theorem

$\exists\, x_0 \in (0,1)$, such that $f(x_0) = 0$.

So has one and only one less than 1 and positive root.

Examples

Prove $x^5 - 5x + 1 = 0$ has one and only one less than 1 and positive root.

(2) Uniqueness

Let another $x_1 \in (0,1), x_1 \neq x_0$, **such that** $f(x_1) = 0$.

$\because$ $f(x)$ satisfying Rolle Theorem between x_0 and x_1.

$\therefore \exists \xi \, (x_0, x_1), \; f'(\xi) = 0.$

but $f'(x) = 5(x^4 - 1)$ **< 0,** $(x \in (0,1))$ Contradict!

$\therefore$ unique real root.

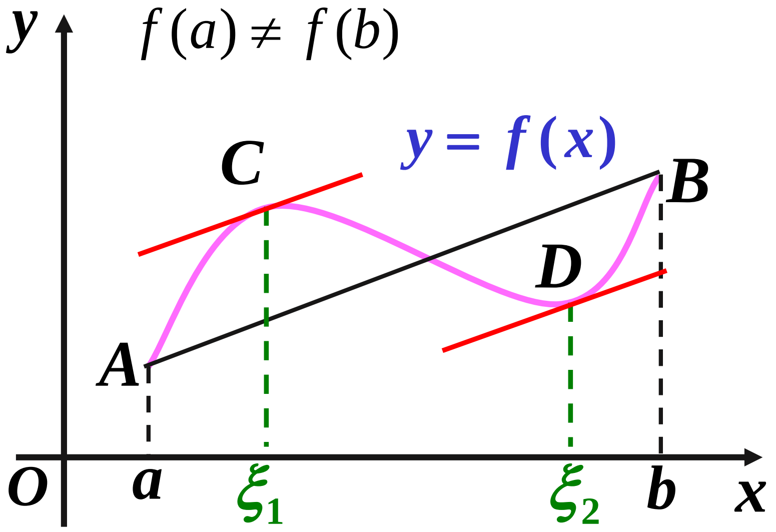
Result: Between the two real roots of f(x)=0, there exist at least one root for $f'(x) = 0$

Theorem of the Mean Value Theorem for Derivatives

Th A: Mean Value Theorem for Derivatives

$$f \in C[a,b], \quad f'(x) \text{ exists in } (a,b)$$

$$\Rightarrow \exists \xi \in (a,b) \text{ s.t. } f'(\xi) = \frac{f(b)-f(a)}{b-a}$$



The Mean Value Theorem for Derivatives

$$g(x) - f(a) = \frac{f(b) - f(a)}{b - a} \bullet (x - a)$$ (the point-slope form)

$$\Rightarrow s(x) = f(x) - g(x) \quad = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \bullet (x - a)$$

$s(a) = s(b) = 0$ (at the two end points)

$$\exists x \in (a, b) \Rightarrow s'(x) = f'(x) - g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow s'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Theorem

Theorem B : Method I

If $F(x)$ is an antiderivative of $f(x)$ on the interval I

Then any antiderivative of $f(x)$ on the interval I can be expressed as

$\boldsymbol{F(x)+C}$,C is any constant.

If $G(x)$ is another antiderivative of $f(x)$, then $\boldsymbol{G'(x)=f(x)}$,

We need to proof : $G(x)-F(x)\equiv$ constant

$\boldsymbol{F'(x)=f(x)}$, and $[G(x)-F(x)]' = G'(x) - F'(x) = f(x)-f(x) \equiv \boldsymbol{0}$

The function whose derivative is always zero must be a constant

So $G(x)-F(x)=C_0 \Rightarrow G(x)=F(x)+C_0$

any constant

Theorem B (Method II, in textbook!)

If $F'(x) = G'(x)$ for all x in (a, b), then there is a constant C such that $F(x) = G(x) + C$ for all x in (a, b).

Let $H(x) = F(x) - G(x)$. Then $H'(x) = F'(x) - G'(x) = 0$ for all x in (a, b).

Choose x_1, as some (fixed) point in (a, b),and let x be any other point there.

The function H satisfies the hypotheses of the Mean Value Theorem on the closed interval with end points x_1 and x.

$\exists c$ between x_1 and x s.t. $H(x) - H(x_1) = H'(c)(x - x_1)$ and $H'(c) = 0$

Therefore, $H(x) - H(x_1) = 0$, Namely, $H(x) = H(x_1)$ for all x in (a, b).

Since $H(x) = F(x) - G(x)$, Namely, $\mathrm{F}(x) - G(x) = H(x_1)$.

Let $C = H(x_1)$, and we have the conclution $F(x) = G(x) + C$.

Example 1

Let $f(x) = x^3 - x^2 - x + 1$ on $[-1,2]$. Find all numbers c satisfying the conclusion to the Mean Value Theorem.

We find $f'(x) = 3x^2 - 2x - 1$

and $\frac{f(2)-f(-1)}{2-(-1)} = \frac{3-0}{3} = 1$

Therefore, $3c^2 - 2c - 1 = 1$

Namely, $3c^2 - 2c - 2 = 0$

By the Quadretic Formula, there are two solutions,

$c = (2 \pm \sqrt{4 + 24})/6$, namely, $c = (1 \pm \sqrt{7})/3$

$c_1 \approx -0.55, c_2 \approx 1.22$, both numbers are in the interval $(-1,2)$.

Example 2

$f(b)-f(a)=f'(\xi)(b-a) \quad \xi\in(a,b)$

Prove that $\arctan x_2 - \arctan x_1 \le x_2 - x_1, \quad (x_1 < x_2).$

Let $f(x)=\arctan x$ on $[x_1, x_2]$

By Mean Value Theorem for Derivatives, we have

$$\arctan x_2 - \arctan x_1 = \frac{1}{1+\xi^2}(x_2 - x_1) \quad \xi\in(x_1, x_2),$$

$$\because \frac{1}{1+\xi^2}\le 1,$$

$$\therefore \arctan x_2 - \arctan x_1 \le x_2 - x_1.$$

Example 3

Prove that $\frac{x}{1+x} < \ln(1+x) < x$, when $x > 0$.

Let $f(x) = \ln(1+x)$, $[0, x]$ important idea

By Mean Value Theorem for Derivative , we have

$$\therefore f(x) - f(0) = f'(\xi)(x-0), \quad (0 < \xi < x)$$

$$\because f(0) = 0, f'(x) = \frac{1}{1+x}, \text{ we have } \ln(1+x) = \frac{x}{1+\xi},$$

By $0 < \xi < x \Rightarrow 1 < 1+\xi < 1+x \Rightarrow \frac{1}{1+x} < \frac{1}{1+\xi} < 1,$

$$\therefore \frac{x}{1+x} < \frac{x}{1+\xi} < x, \text{ namely } \frac{x}{1+x} < \ln(1+x) < x.$$

Summary of the Mean Value Theorem for Derivatives

$\text{Th}: f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$ can be written in other forms: $\xi \in (a, b)$,

(i) $f(b) - f(a) = f'(\xi)(b - a)$. When $a > b$ still holds.

(ii) $f(x + \Delta x) - f(x) = f'(\xi)\Delta x, \ (x < \xi < \Delta x + x)$

(iii) $\Delta y = f'(x + \theta\Delta x) \cdot \Delta x$ $(0 < \theta < 1)$.

accurate expression of increment Δy.

Questions and Answers

Prove $\arcsin x + \arccos x = \dfrac{\pi}{2}\ (-1 \le x \le 1).$

Let $f(x) = \arcsin x + \arccos x, \quad x \in [-1, 1]$

$\because\ f'(x) = \dfrac{1}{\sqrt{1-x^2}} + \left(-\dfrac{1}{\sqrt{1-x^2}}\right) = 0. \ \therefore\ f(x) \equiv C, \quad x \in [-1,1]$

$\because\ f(0) = \arcsin 0 + \arccos 0 = 0 + \dfrac{\pi}{2} = \dfrac{\pi}{2},$

Namely, $C = \dfrac{\pi}{2}$. $\quad \therefore\ \arcsin x + \arccos x = \dfrac{\pi}{2}.$

Prove by yourself: $\operatorname{arctan} x + \operatorname{arccot} x = \dfrac{\pi}{2}, x \in (-\infty, +\infty).$

Questions and Answers

Find c for $f(x) = 2\sqrt{x}$ on $[1,4]$.

$$f'(x) = \frac{1}{\sqrt{x}}$$ and

$$\frac{f(4) - f(1)}{4-1} = \frac{4-2}{3} = \frac{2}{3}$$ Thus, we must have

$$\frac{1}{\sqrt{c}} = \frac{2}{3},\ c \in (1,4)$$

The single solution is $c = \dfrac{9}{4}$.

The Mean Value Theorem for Derivatives